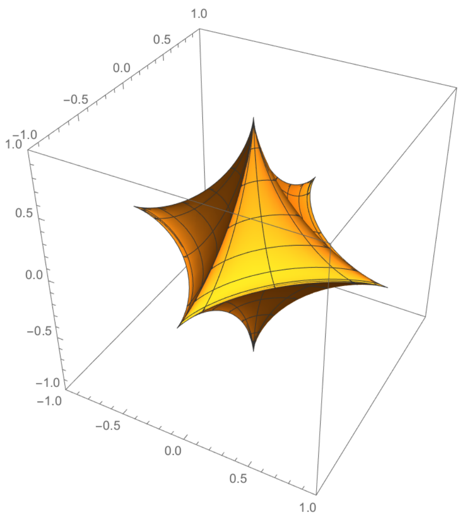

Gaussian Curvature

GCIME

February 25, 2021

General Information/Guidelines

1. DO NOT OPEN THIS BOOKLET UNTIL YOU GIVE THE SIGNAL TO BEGIN.
2. This is a **15**-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators, calculating devices, smart phones or watches,** ***and*** **computers** (other than to access the problems) **are not permitted.**
4. Record all your answers on any answer form or scratch paper. For submitting your answers, send a private message on AoPS to users **Phoenixfire**, **EpicNumberTheory**, **Quantumfluctuations**, **Rama1728** and **Aritra12**.
5. The problems of the GCIME begin on the next page. Good Luck!

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1. Let $\pi(n)$ denote the number of primes less than or equal to n. Suppose $\pi(a)^{\pi(b)} = \pi(b)^{\pi(a)} = c$. For some fixed c what is the maximum possible solutions (a,b,c) but not exceeding 99.

2. Let N denote the number of positive integer solutions of the given equation:
$$\lfloor\sqrt{n}\rfloor + \lfloor\sqrt[3]{n}\rfloor + \lfloor\sqrt[4]{n}\rfloor + \lfloor\sqrt[5]{n}\rfloor = 100.$$
What is the value of N?

3. Let $ABCD$ be a cyclic kite. Let $r \in \mathbb{N}$ be the inradius of $ABCD$. Suppose $AB \cdot BC \cdot r$ is a perfect square. What is the smallest value of $AB \cdot BC \cdot r$?

4. Define $H(m)$ as the harmonic mean of all the divisors of m . Find the positive integer $n < 1000$ for which $\frac{H(n)}{n}$ is the minimum amonst all $1 < n \leq 1000$.

5. Let x be a real number such that
$$\frac{\sin^4 x}{20} + \frac{\cos^4 x}{21} = \frac{1}{41}.$$
If the value of
$$\frac{\sin^6 x}{20^3} + \frac{\cos^6 x}{21^3}$$
can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers, then find the remainder when $m+n$ is divided by 1000.

6. Two scales used to measure temperature are degrees Fahrenheit (F) and degrees Celsius (C) and the two are related by the formula $F = \frac{9}{5}C + 32$. When a two-digit integer degree temperature n in Celcius is converted to Fahrenheit and rounded to the nearest integer degree, it turns out the ones and tens digits of the original Celsius temperature n sometimes switch places to give the rounded Fahrenheit equivalent. Find the sum of all two-digit integer values of n for which this happens.

7. Let a_n denote the units digit of $(4n)^{(3n)^{(2n)^n}}$. Then find the sum of all positive integers $n \leq 1000$ such that
$$\sum_{i=1}^{n} a_i < 4n.$$

8. A basketball club decided to label every basketball in the club. After labelling all n of the balls, the labeller noticed that exactly half of the balls had the digit 1. Find the sum of all possible three-digit integer values of n.

9. $\triangle ABC$ has perimeter 60, and points D, E, and F are chosen on sides BC, AC, and AB respectively. If the circumcircles of triangles $\triangle AFE$, $\triangle BFD$, and $\triangle CED$ all pass through the orthocenter of $\triangle DEF$, then the maximum possible area of $\triangle DEF$ can be written as $a\sqrt{b}$ for squarefree b. What is $a+b$?

10. Let x, y and z be randomly chosen real numbers from the interval $[-10,10]$. Let the probability that these randomly chosen x, y, z satisfy the following inequalities

$$10(|x|+|y|+|z|) \geq 100 \geq x^2+y^2+z^2$$

be $\frac{m}{n}$ where m and n are coprime to each other. Find $m+n$.

11. Let x,y be positive integers such that $\gcd(x,y)=1$ and

$$\tau(x)^{\tau(y)^2} = \tau(y)^{\tau(x)}.$$

Where $\tau(n)$ denotes the number of divisors of n. Find the sum of all possible values of $\tau(xy)$.

12. Let a,b be positive integers. On the real line A stands at $-a$ and B stands at b. A fair coin is tossed, and if it shows heads then A moves one unit to the right, while if it shows tails then B moves one unit to the left. The process stops when A or B reaches the origin. If $E(a,b)$ denotes the expected number of tosses before the process terminates, and the value of $E(5,7)$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers, then find the remainder when $m+n$ is divided by 1000.

13. Find the minimum value of

$$\frac{\sum_{cyc}(\sqrt{a}+c+1)(\sqrt{b}+\sqrt{c}+1)}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^3}.$$

when $a,b,c \in \mathbb{R}^+$ and $\max(a^2+b^2, b^2+c^2, c^2+a^2) \leq 2$.

14. Let

$$f(n) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} (n-r)^n$$

Also, let the value of $f(2021)$ be represented in decimal system be A. Find the no. of zeroes at the end of A.

15. Find the sum of all positive integers fore which one can match each even number from $2,4,\ldots,2n$ with exactly one odd number from $1,3,\ldots,2n-1$ so that the sums of the resulting n pairs are pairwise relatively prime. For example, if $n=3$, a solution is $1+6=7, 2+3=5, 4+5=9$.

The GCIME Solution Pamphlet will be released after the testing period.

The **GCIME**
A program of GC

The GCIME is supported by major contributions from

AMC AIME TEAM & some members from USAMO IMO team

Finally, we thank you for taking this mock. We hope you enjoyed it!